1. a) **Solve:** $\left(1+x^2\right)\frac{dy}{dx}+2xy=2\cos x$

Solution: Given: $\frac{dy}{dx}+\frac{2x}{1+x^2}y=\frac{2\cos x}{1+x^2}$

Here $P=\frac{2x}{1+x^2}$ and $Q=\frac{2\cos x}{1+x^2}$

$\therefore \quad I.F.=\exp\left(\int P dx\right)=\exp\left(\int \frac{2x}{1+x^2}dx\right)=\exp\left[\log\left(1+x^2\right)\right]=1+x^2$

The solution is,

$$y.I.F.=c+\int I.F.\times Q\,dx$$

$$\Rightarrow \quad y.\left(1+x^2\right)=c+\int\left(1+x^2\right)\times\frac{2\cos x}{1+x^2}dx$$

$$\Rightarrow \quad y.\left(1+x^2\right)=c+2\int \cos x\,dx$$

$$\Rightarrow \quad \boxed{y.\left(1+x^2\right)=c+2\sin x}$$

Answer

b) Solve: $x^2p^3 + y(1+x^2y)p^2 + y^3p = 0$, **where** $p = \frac{dy}{dx}$

Solution: Given: $x^2p^3 + y(1+x^2y)p^2 + y^3p = 0$

$$\Rightarrow \quad p\left[x^2p^2 + yp + x^2y^2p + y^3\right] = 0$$

$$\Rightarrow \quad p\left[x^2p(p+y^2) + y(p+y^2)\right] = 0$$

$$\Rightarrow \quad p(p+y^2)(x^2p+y) = 0$$

$\therefore \quad p = 0,\ p + y^2 = 0$ and $x^2p + y = 0$

Now, $p = 0$

$$\Rightarrow \quad \frac{dy}{dx} = 0$$

Integrating on both sides, we get

$$y = c_1 \Rightarrow y - c_1 = 0 \qquad \ldots (1)$$

Now, $p + y^2 = 0$

$$\Rightarrow \quad \frac{dy}{dx} = -y^2$$

$$\Rightarrow \quad -\frac{dy}{y^2} = dx$$

Integrating on both sides, we get

$$\frac{1}{y} = x + c_2 \quad \Rightarrow \frac{1}{y} - x - c_2 = 0 \qquad \ldots (2)$$

and $x^2p + y = 0$

$$\Rightarrow \quad x^2\frac{dy}{dx} + y = 0$$

$$\Rightarrow \quad \frac{dy}{y} = -\frac{dx}{x^2}$$

Integrating on both sides, we get

$$\log y = \frac{1}{x} + c_3$$

$$\Rightarrow \quad \log y - \frac{1}{x} - c_3 = 0 \qquad \ldots (3)$$

The required solution is,

$$\boxed{(y-c)\left(\frac{1}{y} - x - c\right)\left(\log y - \frac{1}{x} - c\right) = 0} \qquad \text{where } c_1 = c_2 = c_3 = c$$

Answer

2. a) Solve: $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = e^x + 2$

Solution: Given $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = e^x + 2$

$\Rightarrow \quad \left(D^3 - 3D^2 + 3D - 1\right)y = e^x + 2 \qquad \text{as } D \equiv \frac{d}{dx}$

The A.E. is

$$m^3 - 3m^2 + 3m - 1 = 0$$

$\Rightarrow \quad (m-1)^3 = 0$

$\Rightarrow \quad m = 1, 1, 1$

The C.F. is

$$C.F. = \left(c_1 + x c_2 + x^2 c_3\right)e^x$$

Now, $\quad P.I. = \frac{1}{(D-1)^3}\left(e^x + 2\right)$

$\Rightarrow \quad = \frac{1}{(D-1)^3}e^x + 2\left[\frac{1}{(D-1)^3}e^{0x}\right]$

$\Rightarrow \quad = e^x\left[\frac{1}{\left(\overline{D+1}-1\right)^3}.1\right] + 2\left[\frac{1}{(0-1)^3}e^{0x}\right]$

$\Rightarrow \quad = e^x\left[\frac{1}{D^3}.1\right] - 2 = e^x\left(\frac{x^3}{6}\right) - 2$

$\Rightarrow \quad P.I. = \frac{x^3e^x}{6} - 2$

The solution is,

$$\boxed{y = \left(c_1 + x c_2 + x^2 c_3\right)e^x + \frac{x^3e^x}{6} - 2}$$

Answer

b) **Solve:** $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=x^2+2\log x$

Solution: Given differential equation is

$$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=x^2+2\log x \quad \ldots (1)$$

This is homogeneous linear differential equation.

So put $x=e^z$

$\Rightarrow \quad z=\log x$

$\Rightarrow \quad \frac{dz}{dx}=\frac{1}{x}$

and $x\frac{d}{dx}\equiv D,\ x^2\frac{d^2}{dx^2}\equiv D(D-1)$ as $D\equiv\frac{d}{dz}$

then equation (1), becomes

$$[D(D-1)-2D-4]y = e^{2z}+2z$$

$$\Rightarrow \quad [D^2-3D-4]y = e^{2z}+2z$$

The A.E. is,

$$m^2-3m-4 = 0$$

$$\Rightarrow \quad (m+1)(m-4) = 0$$

$$\Rightarrow \quad m = -1, 4$$

$$\therefore \quad C.F. = c_1e^{-z}+c_2e^{4z}=c_1x^{-1}+c_2x^4$$

$$P.I.=\frac{1}{D^2-3D-4}e^{2z}+\frac{1}{D^2-3D-4}2z$$

$$=\frac{1}{2^2-3(2)-4}e^{2z}-\frac{1}{4}\left[1-\left(\frac{D^2-3D}{4}\right)\right]^{-1}2z$$

$$=-\frac{1}{6}e^{2z}-\frac{1}{2}\left[1+\left(\frac{D^2-3D}{4}\right)+\ldots\ldots\right]z$$

$$=-\frac{1}{6}e^{2z}-\frac{1}{2}\left[z+\left(\frac{D^2z-3Dz}{4}\right)+\ldots\right]$$

$$=-\frac{1}{6}e^{2z}-\frac{1}{2}\left[z+\frac{1}{4}(0-3)\right]=-\frac{1}{6}e^{2z}-\frac{z}{2}+\frac{3}{8}$$

$$P.I.=-\frac{x^2}{6}-\frac{\log x}{2}+\frac{3}{8}$$

$\therefore$ The required solution is,

$$y=C.F.+P.I.$$

$$\Rightarrow \quad \boxed{y=c_1x^{-1}+c_2x^4-\frac{x^2}{6}-\frac{\log x}{2}+\frac{3}{8}}$$

Answer

3. a) Solve: $\left(1-x^2\right)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=x\left(1-x^2\right)^{3/2}$

Solution: Given differential equation is

$$\frac{d^2y}{dx^2}+\frac{x}{1-x^2}\frac{dy}{dx}-\frac{y}{1-x^2}=x\left(1-x^2\right)^{1/2} \quad \ldots (1)$$

Here, $P=\dfrac{x}{1-x^2}$, $Q=-\dfrac{1}{1-x^2}$ and $R=x\left(1-x^2\right)^{1/2}$

Clearly $P+Qx=\dfrac{x}{1-x^2}+x\left(-\dfrac{1}{1-x^2}\right)=0$

Therefore $y_1=x$, is a part of C.F., then Suppose the complete solution is

$$y=v\,y_1=vx \quad \ldots (2)$$

Where v is a function of x

Since
$$\frac{d^2v}{dx^2}+\left[P+\frac{2}{y_1}\frac{dy_1}{dx}\right]\frac{dv}{dx}=\frac{R}{y_1}$$

$$\frac{d^2v}{dx^2}+\left[\frac{x}{1-x^2}+\frac{2}{x}(1)\right]\frac{dv}{dx}=\frac{x\left(1-x^2\right)^{1/2}}{x}$$

$\Rightarrow$
$$\frac{d^2v}{dx^2}+\left[\frac{x}{1-x^2}+\frac{2}{x}\right]\frac{dv}{dx}=\left(1-x^2\right)^{1/2} \quad \ldots (3)$$

Taking, $z=\dfrac{dv}{dx} \Rightarrow \dfrac{dz}{dx}=\dfrac{d^2v}{dx^2}$

$\therefore$ From Equation (3), we get

$$\frac{dz}{dx}+\left[\frac{x}{1-x^2}+\frac{2}{x}\right]z=\left(1-x^2\right)^{1/2} \quad \ldots (4)$$

This is Linear differential equation of first order.

Here, $P_1=\dfrac{x}{1-x^2}+\dfrac{2}{x}$ and $Q_1=\left(1-x^2\right)^{1/2}$

and $I.F.=e^{\int P\,dx}=e^{\int\left[\frac{x}{1-x^2}+\frac{2}{x}\right]dx}=e^{\left[-\frac{1}{2}\log\left(1-x^2\right)+2\log x\right]}=e^{\log\left(\frac{x^2}{\sqrt{1-x^2}}\right)}=\dfrac{x^2}{\sqrt{1-x^2}}$

The solution of equation (4) is,

$$z.I.F. = c_1 + \int I.F. \times Q_1 \, dx$$

$$\Rightarrow \quad z.\left(\frac{x^2}{\sqrt{1-x^2}}\right) = c_1 + \int \left[\frac{x^2}{\sqrt{1-x^2}} \times \left(1-x^2\right)^{1/2}\right] dx$$

$$\Rightarrow \quad z.\left(\frac{x^2}{\sqrt{1-x^2}}\right) = c_1 + \int x^2 \, dx$$

$$\Rightarrow \quad z.\left(\frac{x^2}{\sqrt{1-x^2}}\right) = c_1 + \frac{x^3}{3}$$

$$\Rightarrow \quad z = c_1\left(\frac{\sqrt{1-x^2}}{x^2}\right) + \frac{1}{3} x \sqrt{1-x^2}$$

Integrating on both sides, we get

$$\log z = -\log x + \log c_1$$

$$\Rightarrow \quad z = \frac{c_1}{x}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{c_1}{x}$$

$$\Rightarrow \quad dv = c_1 \frac{dx}{x}$$

Integrating on both sides, we get

$$v = c_1 \log x + c_2$$

Putting in equation (2), we get

$$\boxed{y = [c_1 \log x + c_2] e^x}$$ **Answer**

b) Solve in series the equation $\left(1+x^2\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$ about the point $x = 0$.

Solution: Given differential equation is,

$$\left(1+x^2\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0 \qquad \text{... (1)}$$

Here, $P_0(x)=1+x^2$

and $P_0(0)=1+0=1\neq 0$

Therefore $x = 0$ is an ordinary singular point of given differential equation.

Suppose the solution is,

$$y=a_0+a_1x+a_2x^2+a_3x^3+... \qquad \text{... (2)}$$

$$\Rightarrow \quad y=\sum_{k=0}^{\infty} a_k\, x^k \qquad \text{... (3)}$$

Differentiating both sides *w.r.t.* x, we get

$$\frac{dy}{dx}=\sum_{k=0}^{\infty} a_k\, k\, x^{k-1}=\sum_{k=0}^{\infty} a_k\, k\, x^{k-1}$$

and
$$\frac{d^2y}{dx^2}=\sum_{k=0}^{\infty} a_k\, k\,(k-1)\,x^{k-2}=\sum_{k=0}^{\infty} a_k\, k\,(k-1)\,x^{k-2}$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation (1), we get

$$\left(1+x^2\right)\sum_{k=0}^{\infty} a_k\, k\,(k-1)\,x^{k-2}+x\sum_{k=0}^{\infty} a_k\, k\, x^{k-1}-\sum_{k=0}^{\infty} a_k\, x^k=0$$

$$\Rightarrow \quad \sum_{k=0}^{\infty} a_k\, k\,(k-1)\,x^{k-2}+\sum_{k=0}^{\infty} a_k\, k\,(k-1)\,x^{k}+\sum_{k=0}^{\infty} a_k\, k\, x^{k}-\sum_{k=0}^{\infty} a_k\, x^k=0 \qquad \text{... (4)}$$

Equating the coefficient of x^0 on both sides in equation (4), we get

$$2(2-1)\,a_2+0+0-a_0=0 \quad \Rightarrow \quad \boxed{a_2=\frac{a_0}{2}}$$

Equating the coefficient of x^1 on both sides in equation (4), we get

$$3\,(3-1)\,a_3+0+a_1-a_1=0 \quad \Rightarrow \quad \boxed{a_3=0}$$

Equating the coefficient of x^2 on both sides in equation (4), we get

$$4\,(4-1)\,a_4+2\,(2-1).a_2+2\,a_2-a_2=0$$

$$\Rightarrow \quad a_4=-\frac{a_2}{4}=-\frac{1}{4}\left(\frac{a_0}{2}\right) \qquad \text{[Putting the value of } a_2\text{]}$$

$$\Rightarrow \quad \boxed{a_4=-\frac{a_0}{8}}$$

Equating the coefficient of x^3 on both sides in equation (4), we get

$$5\,(5-1)\,a_5 + 3\,(3-1).a_3 + 3\,a_3 - a_3 = 0$$

$\Rightarrow$ $$a_5 = -\frac{2\,a_3}{5} = 0$$ [Putting the value of a_3]

Putting the values of a_2, a_3, a_4 and a_5 in equation (2), we get

$$y = a_0 + a_1 x + \left(\frac{a_0}{2}\right) x^2 + 0\,x^3 + \left(-\frac{a_0}{8}\right) x^4 + 0\,x^5 + \ldots$$

$\Rightarrow$ $$\boxed{y = a_0\left(1 + \frac{x^2}{2} - \frac{x^4}{8} + \ldots\right) + a_1\,x}$$

Answer

4. a) Form a partial differential equation by eliminating arbitrary function from $z = f\left(x^2 - y^2\right)$

Solution: Given function is,

$$z = f\left(x^2 - y^2\right) \qquad \text{... (1)}$$

Partially differentiating w.r.t. x and y on both sides, we get

$$\frac{\partial z}{\partial x} = 2x\, f'\left(x^2 - y^2\right) \qquad \text{... (2)}$$

and $$\frac{\partial z}{\partial y} = -2y\, f'\left(x^2 - y^2\right) \quad \text{i.e. } -\frac{1}{2y}\frac{\partial z}{\partial y} = f'\left(x^2 - y^2\right) \qquad \text{... (3)}$$

From (2) and (3), we get

$$\frac{\partial z}{\partial x} = 2x\left[-\frac{1}{2y}\frac{\partial z}{\partial y}\right]$$

$\Rightarrow$ $$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$$ **Answer**

b) Solve the following differential equation

$$\left(x^2-y^2-z^2\right)p+2xq=2xz, \text{ where } p=\frac{\partial z}{\partial x} \text{ and } q=\frac{\partial z}{\partial y}$$

Solution: Given differential equation is

$$(x^2-y^2-z^2)\,p+2\,x\,y\,q=2\,x\,z \qquad \ldots(1)$$

This is Lagrange LPDE.

Here $P=x^2-y^2-z^2$, $Q=2\,x\,y$ and $R=2\,x\,z$

The Lagrange A.E. is

$$\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2\,x\,y}=\frac{dz}{2\,x\,z}$$

Taking the multipliers x, y and z respectively, we get

$$=\frac{xdx+ydy+zdz}{x\left(x^2-y^2-z^2\right)+y(2\,x\,y)+z(2\,x\,z)}$$

$$\Rightarrow \quad =\frac{x\,dx+y\,dy+z\,dz}{x\left(x^2-y^2-z^2+2y^2+2z^2\right)}$$

$$\Rightarrow \quad \frac{x\,dx+y\,dy+z\,dz}{x\left(x^2+y^2+z^2\right)}$$

Taking the ratio's as

$$\frac{x\,dx+y\,dy+z\,dz}{x\left(x^2+y^2+z^2\right)}=\frac{dz}{2\,x\,z}$$

$$\Rightarrow \quad \frac{2x\,dx+2y\,dy+2z\,dz}{x^2+y^2+z^2}=\frac{dz}{z}$$

Integrating on both sides, we get

$$\log\left(x^2+y^2+z^2\right)=\log z+\log c_1$$

$$\Rightarrow \quad \log\left(\frac{x^2+y^2+z^2}{z}\right)=\log c_1$$

$$\Rightarrow \quad \frac{x^2+y^2+z^2}{z}=c_1 \qquad \ldots(2)$$

Taking Last two ratios, we get

$$\frac{dy}{2\,x\,y}=\frac{dz}{2\,x\,z}$$

$$\Rightarrow \quad \frac{dy}{y}=\frac{dz}{z}$$

Integrating on both sides, we get

$$\log y = \log z + \log c_2$$

$$\Rightarrow \quad \log\left(\frac{y}{z}\right) = \log c_2$$

$$\Rightarrow \quad \frac{y}{z} = c_2 \qquad \ldots (3)$$

The General solution of equation (1), we get

$$\boxed{\phi\left[\frac{x^2 + y^2 + z^2}{z}, \frac{y}{z}\right] = 0}$$ **Answer**

5. a) Solve $x^2p^2 + y^2q^2 = 1$, where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$

Solution: Given,

$$x^2p^2 + y^2q^2 = z^2$$

$\Rightarrow$ $$(xp)^2 + (yq)^2 = z^2 \quad \text{....(1)}$$

Putting $Y = \log y \Rightarrow \frac{\partial Y}{\partial y} = \frac{1}{y}$

And $X = \log x \Rightarrow \frac{\partial X}{\partial x} = \frac{1}{x}$

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{1}{x}\frac{\partial z}{\partial X} \Rightarrow xp = \frac{\partial z}{\partial X}$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = \frac{1}{y}\frac{\partial z}{\partial Y} \Rightarrow yq = \frac{\partial z}{\partial Y}$

Putting those values in equation (1) we get,

$$\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = z^2$$

Let $P = \frac{\partial z}{\partial X}$ and $Q = \frac{\partial z}{\partial Y}$

$\Rightarrow$ $$P^2 + Q^2 = 1 \quad \text{... (2)}$$

This is of the form $f(p, q) = 0$ i.e. the standard form I.

Suppose the solution is,

$$z = ax + by + c \quad \text{... (3)}$$

Partially differentiating w.r.t. x and y on both sides, we get

$$\frac{\partial z}{\partial x} = a \Rightarrow P = a \text{ and } \frac{\partial z}{\partial y} = b \Rightarrow Q = b$$

Putting the value of P and Q in equation (2), we get

$$a^2 + b^2 = 1$$

$\Rightarrow$ $$b = \sqrt{1 - a^2}$$

Putting in equation (3), we get

$$\boxed{z = ax + \left(\sqrt{1 - a^2}\right)y + c}$$

Answer

b) Solve the linear partial differential equation $\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x \partial y}+\frac{\partial^2 z}{\partial y^2}=e^{3x+2y}$

Solution: The given Partial differential equation is

$$\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x \partial y}+\frac{\partial^2 z}{\partial y^2}=e^{3x+2y} \qquad \ldots\ldots (1)$$

Suppose, $D \equiv \frac{\partial}{\partial x},\ D' \equiv \frac{\partial}{\partial y}$

From (1), we have $\left(D^2+2DD'+D'^2\right)z=e^{3x+2y}$

The A.E. is,

$$m^2+2m+1=0 \Rightarrow m=-1,\ -1$$

The C.F. is, $C.F.=\phi_1(y-x)+x\phi_2(y-x)$

$$P.I.=\frac{1}{D^2+2DD'+D'^2}e^{3x+2y}=\frac{1}{(3)^2+2(3)(2)+(2)^2}e^{3x+2y}=\frac{e^{3x+2y}}{25}$$

The Complete solution is,

$$\boxed{z=\phi_1(y-x)+x\phi_2(y-x)+\frac{e^{3x+2y}}{25}}$$

Answer

6. a) Show that the following function $u=\frac{1}{2}\log\left(x^2+y^2\right)$ is harmonic and find its harmonic conjugate functions.

Solution: Given: $u=\frac{1}{2}\log\left(x^2+y^2\right)$

Partially differentiate *w.r.t.*, x and y successively, we get

$$\frac{\partial u}{\partial x}=\frac{1}{2}\left[\frac{2x}{x^2+y^2}\right]=\frac{x}{x^2+y^2} \quad \& \quad \frac{\partial^2 u}{\partial x^2}=\frac{\left(x^2+y^2\right)(1)-x(2x)}{\left(x^2+y^2\right)^2}=\frac{y^2-x^2}{\left(x^2+y^2\right)^2} \quad \ldots (1)$$

and $$\frac{\partial u}{\partial y}=\frac{1}{2}\left[\frac{2y}{x^2+y^2}\right]=\frac{y}{x^2+y^2} \quad \& \quad \frac{\partial^2 u}{\partial y^2}=\frac{\left(x^2+y^2\right)(1)-y(2y)}{\left(x^2+y^2\right)^2}=\frac{x^2-y^2}{\left(x^2+y^2\right)^2} \quad \ldots (2)$$

Adding (1) and (2), we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

$\therefore$ u is harmonic function.

To find conjugate of v:

Since $$dv=\left(\frac{\partial v}{\partial x}\right)dx+\left(\frac{\partial v}{\partial y}\right)dy$$

$$\Rightarrow \quad dv=\left(-\frac{\partial u}{\partial y}\right)dx+\left(\frac{\partial u}{\partial x}\right)dy \qquad \text{[CR equation]}$$

Integrating on both sides we get

$$v=\int_{Keeping\ y\ const.}\left(-\frac{\partial u}{\partial y}\right)dx+\int_{independent\ of\ x.}\left(\frac{\partial u}{\partial x}\right)dy+c$$

$$\Rightarrow \quad v=-\int_{Keeping\ y\ const.}\left(\frac{y}{x^2+y^2}\right)dx+\left(No\ term\ free\ from\ x\right)+c$$

$$\Rightarrow \quad v=-\tan^{-1}\left(\frac{x}{y}\right)+c$$

Thus, $\boxed{v=-\tan^{-1}\left(\frac{x}{y}\right)+c}$ **Answer**

b) Determine the analytic function, whose real part is $e^{2x}(x\cos 2y - y\sin 2y)$.

Solution: Given the real part function is $u = e^{2x}(x\cos 2y - y\sin 2y)$ ... (1)

Partially differentiating w.r.t. x and y, we get From (1), we have

$$\frac{\partial u}{\partial x} = e^{2x}(\cos 2y - 0) + 2e^{2x}(x\cos 2y - y\sin 2y) = e^{x}\cos 2y$$

$$\Rightarrow \quad = e^{2x}(\cos 2y + 2x\cos 2y - 2y\sin 2y) = \phi_1(x, y)$$

$$\Rightarrow \quad \phi_1(z, 0) = e^{2z}(1 + 2z)$$

and $$\frac{\partial u}{\partial y} = e^{2x}(-2x\sin 2y - 2y\cos 2y - \sin 2y) = -e^{2x}(2x\sin 2y + 2y\cos 2y + \sin 2y) = \phi_2(x, y)$$

$$\Rightarrow \quad \phi_2(z, 0) = -e^{2z}(0 + 0 + 0) = 0$$

By Milne's Thomson method, we have

$$f(z) = \int \phi_1(z, 0)\, dz - i\int \phi_2(z, 0)\, dz + c$$

$$\Rightarrow \quad = \int e^{2z}(1 + 2z)\, dz - 0 + c = (1 + 2z)\left(\frac{e^{2z}}{2}\right) - 2\left(\frac{e^{2z}}{4}\right) + c$$

$$\Rightarrow \quad = \frac{e^{2z}}{2}(1 + 2z - 1) + c = z e^{2z} + c$$

Thus, $\boxed{f(z) = z e^{2z} + c}$ **Answer**

This is required analytic function.

7. a) Evaluate the following integral using Cauchy-Integral formula $\int_c \frac{4-3z}{z(z-1)(z-2)}dz$, where C is the circle $|z|=\frac{3}{2}$

Solution: Given $I=\int_c \frac{4-3z}{z(z-1)(z-2)}dz$

The pole of integrand is given by,

$$z(z-1)(z-2)=0 \Rightarrow z=0,1,\ 2$$

Now, $z=0 \Rightarrow |z|=|0|=0<\frac{3}{2}$ [Lies within C]

$z=1 \Rightarrow |z|=|1|=1<\frac{3}{2}$ [Lies within C]

and $z=2 \Rightarrow |z|=|2|=2>\frac{3}{2}$ [Out Side of C]

By Cauchy integral formula,

$$\int_c \frac{4-3z}{z(z-1)(z-2)}dz=\int_{c_1}\frac{\frac{4-3z}{(z-1)(z-2)}}{z}dz+\int_{c_2}\frac{\frac{4-3z}{z(z-2)}}{z-1}$$

$\Rightarrow$ $$=2\pi i\left[\frac{4-3z}{(z-1)(z-2)}\right]_{z=0}+2\pi i\left[\frac{4-3z}{z(z-2)}\right]_{z=1}$$

$\Rightarrow$ $$=2\pi i\left[\frac{4-0}{(0-1)(0-2)}\right]+2\pi i\left[\frac{4-3}{1.(1-2)}\right]$$

$\Rightarrow$ $$=4\pi i-2\pi i=2\pi i$$

Thus, $$\boxed{\int_c \frac{4-3z}{z(z-1)(z-2)}dz=2\pi i}$$ **Answer**

b) Evaluate $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ **for the circle** $|z|=1$

Solution: Given, $I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ ... (1)

Let $z = e^{i\theta} \Rightarrow dz = i\,e^{i\theta}\,d\theta$ *i.e.*, $d\theta = \frac{dz}{iz}$

and $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z^2+1}{2z}$

Now, from (1),

$$I = \int_c \frac{\frac{dz}{iz}}{2+\left(\frac{z^2+1}{2z}\right)}$$

$\Rightarrow$ $$= \frac{2}{i}\int_c \frac{dz}{z^2+4z+1}, \text{ where c is } |z|=1 \quad ...(2)$$

Suppose $f(z) = \frac{1}{z^2+4z+1}$

Taking $z^2+4z+1=0$

$\Rightarrow$ $$z = \frac{-4\pm\sqrt{16-4}}{2} = \frac{-4\pm 2\sqrt{3}}{2} = -2\pm\sqrt{3}$$

Say $z = -2+\sqrt{3} = \alpha$ and $z = -2+\sqrt{3} = \beta$

Now, $z = -2+\sqrt{3} \Rightarrow |z| = \left|-2+\sqrt{3}\right| < 1$ [Lies within the circle C]

and $z = -2-\sqrt{3} \Rightarrow |z| = \left|-2-\sqrt{3}\right| > 1$ [Outside the circle C]

Cleary pole $z = -2+\sqrt{3} = \alpha$ within the circle $|z|=1$ with order 1, then the Residues is

$$[\text{Res } f(z)]_{z=\alpha} = \lim_{z\to\alpha}[(z-\alpha)\,f(z)]$$

$\Rightarrow$ $$= \lim_{z\to\alpha}\left[(z-\alpha)\frac{1}{z^2+4z+1}\right] = \lim_{z\to\alpha}\left[(z-\alpha)\frac{1}{(z-\alpha)(z-\beta)}\right]$$

$\Rightarrow$ $$= \lim_{z\to\alpha}\left[\frac{1}{(z-\beta)}\right] = \frac{1}{\alpha-\beta}$$

$\Rightarrow$ $$= \frac{1}{\left(-2+\sqrt{3}\right)-\left(-2-\sqrt{3}\right)} = \frac{1}{2\sqrt{3}}$$

By Cauchy Residues theorem

$$\int_C f(z)\,dz = 2\pi i\,[\text{Sum of Residues of poles which lie within } C]$$

$$\Rightarrow \quad \int_c \frac{dz}{z^2+4z+1} = 2\pi i\left[\frac{1}{2\sqrt{3}}\right] = \frac{\pi i}{\sqrt{3}}$$

Putting in equation (2), we get

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2}{i}\left(\frac{\pi i}{\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}$$

Thus, $\boxed{\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}}$ **Answer**

Solution By Dr. Akhilesh Jain (Corporate Institute of Science of Technology,
Bhopal,)
Mathematics-II (BT202) RGPV Exam May 2019

8. a) If $\overline{F} = 3xy\hat{i} - y^2\hat{j}$, evaluate $\int_C \overline{F}\, d\vec{r}$, where C is the arc of the parabola $y = 2x^2$ from (0, 0) to (1, 2).

Solution: Given $I = \int_C \overline{F}\, d\vec{r}$

and $y = 2x^2$ then $dy = 4x\, dx$

Suppose $\quad \vec{r} = x\hat{i} + y\hat{j}$

$\Rightarrow \quad d\vec{r} = dx\hat{i} + dy\hat{j}$

$\Rightarrow \quad d\vec{r} = dx\,\hat{i} + 4x\, dx\,\hat{j} = (\hat{i} + 4x\hat{j})\, dx \quad \ldots (1)$

Now, $\quad \overline{F}\, d\vec{r} = \left[3x(2x^2)\hat{i} - 4x^4\hat{j}\right](\hat{i} + 4x\hat{j})\, dx$

$\Rightarrow \quad = (6x^3 - 16x^5)\, dx \quad \left[\because y = 2x^2 \Rightarrow y^2 = 4x^4\right]$

$\therefore \quad \int_C \overline{F}\, d\vec{r} = \int_0^1 \left[6x^3 - 16x^5\right] dx$

$\Rightarrow \quad = \left[\frac{3x^4}{2} - \frac{8x^6}{3}\right]_0^1$

$\Rightarrow \quad = \frac{3}{2} - \frac{8}{3} = -\frac{7}{6}$

Thus, $\boxed{\int_C \overline{F}\, d\vec{r} = -\frac{7}{6}}$ **Answer**

b) Evaluate $\iint_S \overline{A}\,\hat{n}\,ds$, where $\overline{A}=\left(x+y^2\right)\hat{i}-2x\hat{j}+2yz\,\hat{k}$ and S is the surface of the plane $2x+y+2z=6$ in the first octant.

Solution: Given the function is,

$$\overline{A}=\left(x+y^2\right)\hat{i}-2x\hat{j}+2yz\,\hat{k}$$

Let $\phi(x, y, z) = 2x + y + 2z - 6$

Now, $gard\phi = \nabla\phi$

$$\Rightarrow \quad =\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(2x+y+2z-6)=2\hat{i}+\hat{j}+2\hat{k}$$

$\therefore$ The unit vector normal to the given surface is

$$\hat{n}=\frac{gard\phi}{|gard\phi|}=\frac{2\hat{i}+\hat{j}+2\hat{k}}{\left|2\hat{i}+\hat{j}+2\hat{k}\right|}=\frac{2\hat{i}+\hat{j}+2\hat{k}}{\sqrt{4+1+4}}=\frac{2\hat{i}+\hat{j}+2\hat{k}}{3}$$

and $$\overline{A}.\hat{n}=\left[\left(x+y^2\right)\hat{i}-2x\hat{j}+2yz\,\hat{k}\right]\left(\frac{2\hat{i}+\hat{j}+2\hat{k}}{3}\right)$$

$$\Rightarrow \quad =\frac{1}{3}\left(2x+2y^2-2x+4yz\right)=\frac{2}{3}\left(y^2+2yz\right)$$

Let R is the projection on yz-plane of given plane $2x + y + 2z = 6$, then

$$\therefore \quad \iint_S \overline{A}.\hat{n}\,ds=\iint_R \frac{\overline{A}.\hat{n}}{\left|\hat{n}.\hat{i}\right|}\,dy\,dz \qquad (1)$$

Now, $\hat{n}.\hat{i}=\left(\frac{2\hat{i}+\hat{j}+2\hat{k}}{3}\right)\left(\hat{i}\right)=\frac{2}{3}$ and $0\le z\le\frac{6-y}{2},\ 0\le y\le 6$

$$\therefore \quad \iint_S \overline{A}.\hat{n}\,ds=\int_0^6\int_0^{\frac{6-y}{2}}\frac{\frac{2}{3}\left(y^2+2yz\right)}{\frac{2}{3}}\,dy\,dz$$

$$\Rightarrow \quad =\int_0^6\int_0^{\frac{6-y}{2}}\left(y^2+2yz\right)dy\,dz$$

$$\Rightarrow \quad =\int_0^6\left[y^2z+yz^2\right]_0^{\frac{6-y}{2}}dy=\int_0^6\left[y^2\left(\frac{6-y}{2}\right)+y\left(\frac{6-y}{2}\right)^2-0\right]dy$$

$$\Rightarrow \quad =\int_0^6\left(\frac{6-y}{2}\right)y\left[y+\frac{6-y}{2}\right]dy=\int_0^6\left(\frac{6-y}{2}\right)y\left[\frac{6+y}{2}\right]dy$$

$$\Rightarrow \quad =\frac{1}{4}\int_0^6\left(36y-y^3\right)dy=\frac{1}{4}\left[36\left(\frac{y^2}{2}\right)-\frac{y^4}{4}\right]_0^6$$

$$\Rightarrow \quad =\frac{1}{4}\left[18(36-0)-\frac{1}{3}(1296-0)\right]=\frac{1}{4}\left[648-324\right]=81$$

Thus $$\boxed{\iint_S \overline{A}.\hat{n}\,ds=81}$$ **Answer**